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LETTER TO THE EDITOR

Quantum correlations and Nash equilibria of a bi-matrix game

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Abstract
Playing a symmetric bi-matrix game is usually physical implemented by sharing pairs of 'objects' between two players. A new setting is proposed that explicitly shows effects of quantum correlations between the pairs on the structure of payoff relations and the 'solutions' of the game. The setting allows a re-expression of the game such that the players play the classical game when their moves are performed on pairs of objects having correlations that satisfy Bell's inequalities. If players receive pairs having quantum correlations the resulting game *cannot* be considered *another* classical symmetric bi-matrix game. Also the Nash equilibria of the game are found to be decided by the nature of the correlations.

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1. Introduction

Playing a game requires resources for its physical implementation. For example, to play a bi-matrix game the resources may consist of pairs of two-valued 'objects', like coins, distributed between the players. The players perform their *moves* on the objects and later a referee decides payoffs after observing them. Game theory usually links players' actions *directly* to their payoffs, without a reference to the nature of the objects on which the players have made their moves. Analysis of quantum games [1] suggests that radically different 'solutions' can emerge when the *same* game is physically implemented on distributed objects which are quantum mechanically correlated.

Much of recent work on quantum games [1–4] uses a particular quantization scheme [1] developed for a bi-matrix game where two players, on receiving an entangled two-qubit state, play their moves by local and unitary actions on the state. After disentanglement, a measurement of the state rewards the players their payoffs. The payoffs become classical when the moves are performed on a product state. For example, in Prisoners' Dilemma a new and more beneficial equilibrium emerges in its quantum form [1] when the allowed moves are a *chosen* subset of all possible unitary actions [5]. Extending the set of moves to all possible unitary actions results in no equilibrium at all.

Recently Enk and Pike [6] have argued that the emergence of a new equilibrium in quantum Prisoners' Dilemma can also be understood as an equilibrium in a *modified* form of the game. They constructed *another* matrix game, in which players have access to three pure classical strategies instead of two, claiming that it 'captures' everything quantum Prisoners' Dilemma has to offer. Constructing an extended matrix with an extra pure move, in their view, is justified because *also* in quantum Prisoners' Dilemma players can play moves which are superpositions of the two classical moves. Quantum Prisoners' Dilemma can therefore be thought of as being equivalent to playing a modified classical game with an extended matrix.

Truly quantum pairs of objects possess non-local correlations. Though it is impossible to have a local model, producing *exactly* the same data, of a quantum game set-up, how such unusual correlations may *explicitly* affect solutions of a game when implemented with quantum objects? To how far extent solutions of a quantum game themselves can be called 'truly quantum' in nature. To address these questions and to find a quantum game for which it becomes difficult, if not impossible, to construct another classical game the following two *constraints* are suggested [7] that a quantization scheme should follow:

(C1) In both classical and quantum version of the game the *same* set of moves should be made available to the players.

(C2) The players agree, once and for all, on explicit expressions for their payoffs which *must* not be modified when introducing the quantized version of the game.

Only the nature of correlations existing between the objects the players receive will now decide whether the resulting game is *classical* or *quantum*. The idea of a 'correlation game' [7], created to satisfy the constraints C1 and C2, introduces a new set-up to play bi-matrix games. Its motivation comes from EPR-type experiments on singlet states involving correlations of the measurement outcomes. In such experiments Bell's inequalities [9] are known to be the constraints, derived under the *principle of local causes*, on correlations of measurement outcomes of two-valued (dichotomic) variables. Truly quantum correlations are non-local in character and violate the inequalities.

In a quantization scheme, that exploits correlations, players receiving pairs having local correlations, that do not violate the Bell's inequalties, *must* result in their payoffs being classical. As pointed out in [7], despite explicit dependence of the players' payoffs on correlations, quantum payoffs can *still* be obtained in a correlation game *even* when the correlations do not violate Bell's inequalities. In a sense this aspect weakens the argument for a correlation game. In the present paper we try to address this difficulty by following a different approach in re-defining payoff relations in terms of the correlations. The new approach is not faced with the indicated difficulty; i.e. local correlations, that do not violate Bell's inequalities, *always* result in the classical game. Also in the new approach non-local, and truly quantum, correlations result in a game that *cannot* be considered as just *another* classical symmetric bi-matrix game.

2. Classical symmetric bi-matrix games

A symmetric bi-matrix game between two players, Alice and Bob, has the following matrix representation [8]:

$$\text{Alice}\begin{array}{c} \\ \\ S_1 \\ S_2 \end{array}\begin{array}{cc} \text{Bob} & \\ S_1 & S_2 \\ (r,r) & (s,t) \\ (t,s) & (u,u) \end{array} \tag{1}$$

where, for example, Alice and Bob get payoffs s and t, respectively, when Alice plays S_1 and Bob plays S_2. When Alice and Bob play a bi-matrix game their moves consist of deciding the probabilities p and q, respectively, of playing the first strategy S_1. The second strategy S_2 is then played with probabilities $(1-p)$ and $(1-q)$ respectively. The mixed strategy payoffs for the players can be written as [7]

$$P_A(p,q) = Kpq + Lp + Mq + N \qquad P_B(p,q) = Kpq + Mp + Lq + N \tag{2}$$

where the constants K, L, M and N can be found in terms of r, s, t and u, the coefficients of the bi-matrix. A Nash equilibrium (NE) is a pair (p^*, q^*) defined by the following inequalities:

$$P_A(p^*,q^*) - P_A(p,q^*) \geqslant 0 \qquad P_B(p^*,q^*) - P_B(p^*,q) \geqslant 0. \tag{3}$$

For example in the bi-matrix game of Prisoners' Dilemma,

$$\text{Alice}\begin{array}{c} \\ \\ C \\ D \end{array}\begin{array}{cc} \text{Bob} & \\ C & D \\ (3,3) & (0,5) \\ (5,0) & (1,1) \end{array} \tag{4}$$

where C and D represent the strategies of Cooperation and Defection, respectively, the equilibrium-defining inequalities (3) for the matrix (4) are written as

$$(p^*-p)(1+q^*) \leqslant 0 \qquad (q^*-q)(1+p^*) \leqslant 0 \tag{5}$$

giving $p^* = q^* = 0$ or (D, D) as the unique equilibrium.

3. Quantum correlation games (QCGs)

The correlation game [7] uses an EPR-type setting to play a bi-matrix game. Repeated measurements are performed on correlated pairs of objects by two players, each receiving one half. Players Alice and Bob share a Cartesian coordinate system between them and each player's move consists of deciding a direction in a given plane. For Alice and Bob these are the x–z and y–z planes respectively. Call α and β the unit vectors representing the players' moves. Both players have a choice between two different orientations, i.e. α and z for Alice and β and z for Bob. Each player measures the angular momentum or spin of his/her respective half in one of the two directions. Let the vectors α and β make angles θ_A and θ_B, respectively, with the z-axis. To link the players' moves, represented now by angles θ_A and θ_B, to the usual probabilities p and q appearing in a bi-matrix game, an invertible function g is made public at the start of a game. The g-function maps $[0, \pi]$ to $[0, 1]$ and allows us to translate the players' moves to the probabilities p and q.

We assume the results of the measurements are dichotomic variables, i.e. they may take only the values ± 1, and are represented by a, b and c for the directions α, β and the z-axis, respectively. Correlations $\langle ac\rangle$, $\langle cb\rangle$ and $\langle ab\rangle$ can be found from the measurement outcomes, where the two entries in a bracket represent the players' chosen directions. When the z-axis is shared between the players as the common direction, the Bell's inequality[1] is written [9] as

$$|\langle ab\rangle - \langle ac\rangle| \leqslant 1 - \langle bc\rangle. \tag{6}$$

[1] For perfectly anticorrelated pairs the right hand side of the inequality is $1 + \langle bc\rangle$.

The classical correlations, written in terms of θ_A and θ_B, are known to be invertible. This fact allows us to express θ_A and θ_B in terms of the correlations $\langle ac \rangle$ and $\langle cb \rangle$. The g-function then makes possible the translation of θ_A and θ_B into p and q, respectively. In effect the classical bi-matrix payoffs are re-expressed in terms of the classical correlations $\langle ac \rangle$ and $\langle cb \rangle$. We claim now that our classical game is given, *by definition*, in terms of the correlations. Such re-expression opens the way to find 'quantum' payoffs when the correlations become quantum mechanical.

In this setting the players' payoffs involve only the correlations $\langle ac \rangle$ and $\langle cb \rangle$, instead of the three correlations $\langle ac \rangle$, $\langle cb \rangle$ and $\langle ab \rangle$ present in the inequality (6), when the z-axis is the common direction between the players. This aspect results in obtaining 'quantum' payoffs even when the correlations are local and satisfy the inequality (6). The motivation for introducing EPR-type setting to bi-matrix games is to exploit quantum correlations to generate quantum payoffs. So that, when the correlations are local, the classical game must be produced. We show below the possibility of such a connection by some modifications in the setting of a correlation game suggested previously. In the modified setting the classical payoffs are *always* obtained whenever the correlations $\langle ac \rangle$, $\langle cb \rangle$ and $\langle ab \rangle$ satisfy Bell's inequality (6).

4. A new approach towards defining a correlation game

The following modifications are suggested in the setting of a correlation game:

1. A player's move consists of defining a direction in space by orientating a unit vector. However, this direction is not confined to only the x–z or y–z planes. A player's choice of a direction can be *anywhere* in three-dimensional space. Therefore, Alice's move is to define a unit vector α and, similarly, Bob's move is to define a unit vector β.
2. The z-axis is shared between the players as the common direction.
3. On receiving a half of a correlated pair, a player measures its spin in two directions. For Alice these directions are α and z and for Bob these directions are β and z.
4. Each player measures spin with *equal* probability in his or her two directions.
5. Players agree together on *explicit expressions* giving their payoffs P_A and P_B in terms of all three correlations, i.e.

$$P_A = P_A(\langle ac \rangle, \langle cb \rangle, \langle ab \rangle) \qquad P_A = P_A(\langle ac \rangle, \langle cb \rangle, \langle ab \rangle). \tag{7}$$

These modifications eliminate the need for introducing the g-functions as done in [7]. The modifications are also consistent with the constraints C1 and C2 and the idea of a correlation game developed in [7] essentially retains its spirit. More importantly, a player's move can be in *any* direction in space.

4.1. Defining correlation payoffs in the new approach

A possible way is shown now to define the correlation payoffs (7) which reduce to the classical payoffs (2) whenever the correlations $\langle ab \rangle$, $\langle ac \rangle$ and $\langle bc \rangle$ satisfy the inequality (6).

Consider two quantities ε and σ defined as follows:

$$\varepsilon = \sqrt{3 + \langle bc \rangle^2 + 2\langle ab \rangle \langle ac \rangle} \qquad \sigma = \sqrt{2(1 + \langle bc \rangle) + \langle ab \rangle^2 + \langle ac \rangle^2}. \tag{8}$$

The quantities ε and σ can adapt only real values because the correlations $\langle ac \rangle$, $\langle cb \rangle$ and $\langle ab \rangle$ are always in the interval $[-1, 1]$. Consider now the quantities $(\varepsilon - \sigma)$ and $(\varepsilon + \sigma)$. By definition ε and σ are non-negative, therefore, the quantity $(\varepsilon + \sigma)$ always remains non-negative. It is observed that if $0 \leqslant (\varepsilon - \sigma)$ then the correlations $\langle ac \rangle$, $\langle cb \rangle$ and $\langle ab \rangle$ satisfy the inequality (6). That is because if $0 \leqslant (\varepsilon - \sigma)$ then $0 \leqslant (\varepsilon + \sigma)(\varepsilon - \sigma) = \varepsilon^2 - \sigma^2$. But

$\varepsilon^2 - \sigma^2 = (1 - \langle bc \rangle)^2 - |\langle ab \rangle - \langle ac \rangle|^2$ so that $|\langle ab \rangle - \langle ac \rangle|^2 \leqslant (1 - \langle bc \rangle)^2$ which results in the inequality (6). All the steps in the proof can be reversed and it follows that when the correlations $\langle ac \rangle$, $\langle cb \rangle$ and $\langle ab \rangle$ satisfy Bell's inequality, the quantity $(\varepsilon - \sigma)$ remains non-negative.

For a singlet state satisfying the inequality (6) both the quantities $(\varepsilon + \sigma)$ and $(\varepsilon - \sigma)$ are non-negative and must have maxima. Hence, it is possible to find two non-negative numbers $\frac{(\varepsilon - \sigma)}{\max(\varepsilon - \sigma)}$ and $\frac{(\varepsilon + \sigma)}{\max(\varepsilon + \sigma)}$ in the range [0, 1], whenever the inequality (6) holds. Because $0 \leqslant \varepsilon, \sigma \leqslant \sqrt{6}$ we have $\max(\varepsilon - \sigma) = \sqrt{6}$ and $\max(\varepsilon + \sigma) = 2\sqrt{6}$. The numbers $(\varepsilon - \sigma)/\sqrt{6}$ and $(\varepsilon + \sigma)/2\sqrt{6}$ are in the range [0, 1] when the inequality holds. These numbers are also *independent* of each other.

The above argument paves the way to associate a pair (p, q) of independent numbers with the players' moves (α, β), that is

$$p = p(\alpha, \beta) \qquad q = q(\alpha, \beta) \tag{9}$$

where p, q are in the interval [0, 1] for all directions α, β, when the input states do not violate the inequality (6). From the pair (p, q) a directional pair can also be found as

$$\alpha = \alpha(p, q) \qquad \beta = \beta(p, q) \tag{10}$$

but more than one pair (α, β) of directions may correspond to a given pair of numbers. The converse, however, is not true for known input states. That is, for known input states, only one pair (p, q) can be obtained from a given pair (α, β) of directions.

Players' payoffs can now be expressed in a *correlation form* by the following replacements,

$$p(\alpha, \beta) \sim (\varepsilon - \sigma)/\sqrt{6} \qquad q(\alpha, \beta) \sim (\varepsilon + \sigma)/2\sqrt{6} \tag{11}$$

leading to a re-expression of the classical payoffs (2) as

$$\begin{aligned} P_A(\alpha, \beta) &= Kp(\alpha, \beta)q(\alpha, \beta) + Lp(\alpha, \beta) + Mq(\alpha, \beta) + N \\ P_B(\alpha, \beta) &= Kp(\alpha, \beta)q(\alpha, \beta) + Mp(\alpha, \beta) + Lq(\alpha, \beta) + N \end{aligned} \tag{12}$$

or more explicitly as

$$\begin{aligned} P_A(\alpha, \beta) &= \frac{K}{12}(\varepsilon^2 - \sigma^2) + \frac{L}{\sqrt{6}}(\varepsilon - \sigma) + \frac{M}{2\sqrt{6}}(\varepsilon + \sigma) + N \\ P_B(\alpha, \beta) &= \frac{K}{12}(\varepsilon^2 - \sigma^2) + \frac{M}{\sqrt{6}}(\varepsilon - \sigma) + \frac{L}{2\sqrt{6}}(\varepsilon + \sigma) + N \end{aligned} \tag{13}$$

where a player's payoff now depends on the direction s/he has chosen. The payoffs (13) are obtained under the constraints C1 and C2 and are now functions of all the three correlations.

Relations (9) can also be imagined as follows. When Alice decides a direction α in space, it corresponds to a curve in the p–q plane. Similarly, Bob's decision of the direction β defines another curve in the p–q plane. Relations (11) assure that only one pair (p, q) can then be obtained as the intersection between the two curves.

The set-up assures that for input states satisfying the inequality (6), all of the players' moves (α, β) result in the correlation payoffs (13) generating identical to the classical payoffs (2). For such input states relations (11) give the numbers p, q in the interval [0, 1], which can then be interpreted as probabilities. However, for input states violating the inequality (6), a pair $(p, q) \in [0, 1]$ *cannot* be associated with players' moves (α, β). It is because for such states the quantity $(\varepsilon - \sigma)$ becomes negative and the correlation payoffs (13) generate results having a different form from the classical payoffs (2).

5. Nash equilibria of QCGs

Because the players' moves consist of defining directions in space, the Nash inequalities can be written as

$$P_A(\alpha_0, \beta_0) - P_A(\alpha, \beta_0) \geqslant 0 \qquad P_B(\alpha_0, \beta_0) - P_A(\alpha_0, \beta) \geqslant 0 \tag{14}$$

where the pair (α_0, β_0) corresponds to the pair (p^*, q^*), defined in equation (3), via relations (11). The inequalities (14) are same as the inequalities (3), except their re-expression in terms of the directions.

When the correlations in the input state violate the inequality (6), the payoff relations (13) also lead to disappearance of the classical equilibria. It can be seen, for example, by considering the Nash inequalities for the Prisoners' Dilemma (5). Let the directional pair (α_D, β_D) correspond to the equilibrium (D, D), that is, the inequalities (14) are written as

$$P_A(\alpha_D, \beta_D) - P_A(\alpha, \beta_D) \geqslant 0 \qquad P_B(\alpha_D, \beta_D) - P_A(\alpha_D, \beta) \geqslant 0. \tag{15}$$

Assume the players receive input states that violate the inequality (6). It makes the quantity $(\varepsilon - \sigma) < 0$, that is, the players' moves α and β will not correspond to a point in the *p*–*q* plane where $p, q \in [0, 1]$. Also the directional pair (α_D, β_D) does not remain a NE. It is because the pair (α_D, β_D) is a NE *only* if players' choices of *any* directional pair (α, β) correspond to a point in the *p*–*q* plane where $p, q \in [0, 1]$. Because for input states that violate the inequality (6) a pair of players' moves (α, β) does not correspond to a point in the *p*–*q* plane with $p, q \in [0, 1]$, hence, the directional pair (α_D, β_D) does not remain a NE in the quantum game. The disappearance of the classical equilibrium now becomes *linked* with the violation of the inequality (6) by the correlations in the input states.

6. Quantum game as another classical game?

Coming back to the questions raised in the introduction, we now try to construct a classical bi-matrix game, corresponding to a quantum game, resulting from the payoff relations (13). The classical game is assumed to have the *same* general structure of players' payoffs as given in equations (2). This assumption derives from the hope that the quantum game, corresponding to correlations in the input states that violate the inequality (6), is also equivalent to *another* symmetric bi-matrix game. It is shown below that such a construction cannot be permitted.

Suppose the input states violate the inequality (6). For *any* direction Alice chooses to play, her payoff given by equations (13) can also be written as

$$P_A(\alpha, \beta) = K' pq + L' p + Mq + N \tag{16}$$

where $K' = -K$ and $L' = -L$ and $p, q \in [0, 1]$. Assuming that the constants K', L', M and N define a 'new' symmetric bi-matrix game, Bob's payoff should then be written as

$$P_B(p, q) = K' pq + Mp + L' q + N. \tag{17}$$

But in fact (17) is not obtained as Bob's payoff in the quantum game with correlations violating the inequality (6). Bob's payoff in the quantum game is given as

$$P_B(p, q) = K' pq + M' p + Lq + N \tag{18}$$

where $M' = -M$. Hence the game resulting from the presence of quantum correlations in the input states *cannot* simply be explained as another classical symmetric bi-matrix game: a game obtained by defining new coefficients of the matrix involved. Players' payoffs in the quantum game reside outside the structure of payoffs of a classical symmetric bi-matrix game. The payoffs can be explained within this structure *only* by invoking negative probabilities.

An asymmetric bi-matrix game can, of course, be constructed having identical solutions to the quantum game. In fact for *any* quantum game a classical model can *always* be constructed that summarizes the complete situation and has solutions identical to the quantum solutions as far as the players' payoffs are concerned—a model that relates players' moves directly to their payoffs in accordance with the usual approach in game theory. But still it is not an answer to our initial question: how solutions of a game are affected by the presence of quantum correlations between the physical objects used to implement the game? It is because the question can then simply be rephrased as: what if the modified classical game is played with physical objects having quantum correlations?

7. Summary

The idea of a correlation game is about re-expression of payoffs of a classical bi-matrix game in terms of correlations of measurement outcomes made on pairs of correlated particles. The measurement outcomes are dichotomic variables and their correlations are obtained by averaging over a large number of pairs. Bell's inequalities represent constraints on these correlations obtained under the principle of local causes. A re-expression of the classical payoffs of a bi-matrix game in terms of correlations opens the way to explicitly see the effects of quantum correlations on the solutions of the game.

In this paper a new setting is proposed where two players play a bi-matrix game by repeatedly performing measurements on correlated pairs of objects. The setting is motivated by EPR-type experiments performed on singlet states. On receiving a half of a pair, a player makes a measurement of its spin in one of the two directions available to him/her. The measurements are performed with *equal probability* in the two directions. Both players share a common direction and defining the *other* direction is a player's *move*.

We show how within this set-up a correlation version of a symmetric bi-matrix game can be defined. The correlation game shows some interesting properties. For example, it reduces to the corresponding classical game when the correlations in the input states are local and do not violate Bell's inequality (6). However, when the inequality is violated, the stronger correlations generate results that can be understood, within the structure of classical payoffs in a symmetric bi-matrix game, *only* by invoking negative probabilities. It is shown that a classical Nash equilibrium is affected when the game is played with input states having quantum correlations. The proposed set-up also provides a new perspective on the possibility of reformulating Bell's inequalities in terms of a bi-matrix game played between two spatially-separated players.

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